Integrals
By Professor Willian Neris

In collaboration with
Josue Arreaga
Teacher Assistant

May 12, 2019

1 Integrals

We begin this chapter by introducing the idea of an antiderivative. An antiderivative deals with the connection between derivatives, and the area under the curve. For instance, in the **introduction to derivative** packet we covered how how displacement, velocity, and acceleration are connected. Each of them is the derivative of the previous. With antiderivatives (integrals) we can now use acceleration (the second derivative) to find the original function $f(x)$. Furthermore, we can also find the displacement of the original formula by computing an integral between two given points.

1.1 Definition 1 - antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

1.2 Theorem - 1

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

1.3 Definition 2

The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)dx.$$

The symbol $\int$ is an **integral sign.** The function f is the **integrand** of the integral, and x is the **variable of integration**.

1.3.1 Example of an indefinite Integral

Suppose we are given the function of velocity represented by $f'(x) = 3x^4 + 5x^2 + 4x + 3$ and we want to find the original function. We can do this using an **indefinite integral**(an integral not defined between two points). Basically we are working backwards, in this case think *what do we need to derive in order to get* $f'(x)$.

$$\int (3x^4 + 5x^2 + 4x + 3)dx$$

This is then equal to

$$= \frac{3x^5}{5} + \frac{5x^3}{3} + \frac{4x^2}{2} + 3x + C \text{ (C is a constant).}$$

Now, we check our work by taking the derivative of this function $f(x)$.

$$\frac{d}{dx}(\frac{3x^5}{5} + \frac{5x^3}{3} + \frac{4x^2}{2} + 3x + C)$$

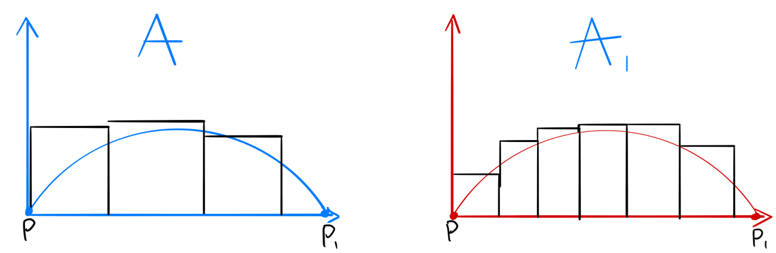
This gives us back the integral

$$f'(x) = (3x^4 + 5x^2 + 4x + 3)$$

We will more in depth on tricks of basic integration and the patterns utilized. For now however, look at the exercise and try to figure it out. Can you see the pattern?

1.4 Difference between Definite and Indefinite integrals.

So far we have introduced the idea of indefinite integrals and how they are interconnected between acceleration, velocity, and the original displacement function. For instance, the integral of acceleration is velocity and the integral of velocity is the displacement function. Indefinite integrals however, do not help us when it comes to finding the area under a graph (this is equal to the displacement). In order for us to find the area under any graph we must use points through which we calculate the area (displacement). This takes the form of $\int_a^b f(x)dx$. We will now show in a more geometric way what an integral is.



Notice graph A and graph A_1, both are identical. They each have the same $f(x)$ function and the same two points P and P_1. In order to calculate their area we will use something known as Riemann sums. A Riemann sum splits the figure into rectangles we can easily calculate (recall the area of a rectangle is base times height). In graph A we use three rectangles and in graph A_1 we use 6 rectangles. Which figure do you think is closest to representing the accurate area of the figure? The answer is graph A_1. Basically the more rectangles we use the better we approach the actual area of the curve. If we used a huge enough number 100, 1000, 1000... the rectangles will eventually approach the real area of the function. This is what an integral does for us. By integrating a function between two points we are basically using a infinity amount of rectangles which helps us approximate the area of the graph accurately. We will now introduce the formal definition of definite integrals, integrability of continuous functions and some algebraic rules. The main idea a reader should keep however is the one we explained.

1.5 Definition 3 - Definite Integral of F over [a,b]

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number j is the **definite integral of f over** $[a, b]$ and that J is the limit of the Riemann sums $\sum_{k=1}^{1} f(c_k)\Delta x_k$ if the following condition is satisfied: Given any $e > 0$ there is a corresponding $\delta \ > 0$ such that for every partition $P = x_0, x_1 \ldots, x_n$ of $[a, b]$ with $||P|| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \ | < e$$

1.6 Theorem 2 - Integrability of Continuous Functions

If a function f is continuous over the interval; $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\displaystyle\int_a^b f(x)dx$ exists and f is integrable over $[a, b]$.

1.7 Theorem 3 - Algebraic Properties of Integrals

When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the following rules

1. Order of Integration: $\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. Zero Width Interval: $\int_b^a f(x)dx = 0$
3. Constant Multiple: $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
4. Sum and Difference: $\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. Additivity: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
6. Max-Min Inequality: If f has maximum value max f and minimum value min f on $[a, b]$, then

$$(\min\ f)\cdot(b-a) \leq \int_a^b f(x)dx \leq (\max\ f)\cdot(b-a).$$

7. Domination: If $f(x) \geq g(x)$ on $[a, b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

$$\text{If } f(x) \geq 0 \text{ on } [a, b] \text{ then } \int_a^b f(x)dx \geq 0.$$

1.8 Definition 4 - Area Under the curve

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b,

$$\text{A} = \int_a^b f(x)dx.$$

1.9 Definition 5 - The Mean

If f is integrable $[a, b]$, then its average value on $[a, b]$, which is also called its **mean**, is

$$av(f) = \tfrac{1}{b-a}\int_a^b f(x)dx.$$

1.10 Theorem 4 - Mean Value Theorem for Definite Integrals

if f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \tfrac{1}{b-a}\int_a^b f(x)dx.$$

Proof If we divide both sides of the Max - Min Inequality (Rule 6, Theorem 23) by $(b-a)$, we obtain

$$\min f \leq \tfrac{1}{b-a}\int_a^b f(x)dx \leq \max\ f.$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions says that f must assume every value between min f and max f. IT must therefore assume the value $(1/(b-a))\int_a^b f(x)dx$ at some point c in $[a, b]$.

The continuity of f is important here. It is impossible for a discontinuous function to never equal its average value.

1.11 Theorem 5 - Fundamental Theorem of Calculus, Part 1

if f is continuous on $[a, b]$, then $F(x) = \displaystyle\int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \tfrac{d}{dx} \int_a^x f(t)dt = f(x).$$

Proof We prove the Fundamental Theorem, Part 1, by applying the definition of the derivative directly to the function $F(x)$, when x and $x + h$ are in (a, b). This means writing out the difference quotient

$$\tfrac{F(x+h)-F(x)}{h}$$

and showing that its limit as $h \to 0$ is the number $f(x)$. Doing so, we find that

$$F'(x) = \lim_{h\to 0} \tfrac{F(x+h)-F(x)}{h}$$

$$= \lim_{h\to 0} \tfrac{1}{h}[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt]$$

$$= \lim_{h\to 0} \tfrac{1}{h} \int_x^{x+h} f(t)dt.$$

According to the Mean Value Theorem for Definite Integrals, there is some point c between x and $x+h$ where $f(c)$ equals the average value of f on the interval $[x, x + h]$. That is, there is some number c in $[x, x + h]$ such that

$$\tfrac{1}{h} \tfrac{x}{x+h} f(t)dt = f(c).$$

As $h \to 0, x + h$ approaches x which forces c to approach x also (because c is trapped between x and $x + h$). Since f is continuous at x, $f(c)$ therefore approaches $f(x)$:

$$\lim_{h\to 0} f(c) = f(x).$$

Hence we have shown that, for any x in (a, b),

$$F'(x) = \lim_{h\to 0} \tfrac{1}{h} \int_x^{x+h} f(t)dt$$

$$= \lim_{h\to 0} f(c)$$

$$f(x),$$

and therefore F is differentiable at x. Since differentiability implies continuity, this also shows that F is continuous on the open interval (a, b). To complete the proof, we just have to show that F is also continuous at $x = a$ and $a = b$. To do this, we make a very similar argument, except that $x = a$ we need only consider the one-sided limit as $h \to 0^+$, and similarly at $x = b$ we need only consider $h \to 0^-$. This shows that F has a one-sided derivative at $x = a$ and $x = b$, and therefore Theorem 1 of the derivatives packet implies that F is continuous at those points.

1.12 Theorem 5 - Fundamental Theorem of Calculus, Part 2

If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof Part 1 of the Fundamental Theorem tells us that an antiderivative of f exists, namely

$$G(x) = \int_a^x f(t)dt.$$

Thus, F is *any* antiderivative of f, then $F(x) = G(x) + C$ for some constant C for $a < x < b$ Since both F and G are continuous on $[a, b]$, we see that the equality $F(x) = G(x) + C$ also holds when $x = a$ and $x = b$ by taking one-sided limits (as $x \to a^+$ and $x \to b^-$).

Evaluating $F(b) - F(a)$, we have

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)dt - \int_a^a f(t)dt \\ &= \int_a^b f(t)dt - 0 \\ &= \int_a^b f(t)dt. \end{aligned}$$

1.13 Theorem 6 - The Net Change

The net change in a differentiable function $F(x)$ over an interval $a \leq x \leq b$ is the integral of its rate of change:

$$\text{F(b) - F(a)} = \int_a^b F'(x)dx.$$

1.14 Theorem 7 - The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du$$

Proof By the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f, because

$$\begin{aligned} \tfrac{d}{dx} F(g(x)) &= F'(g(x)) \cdot g'(x) \\ &= f(g(x)) \cdot g'(x) \end{aligned}$$

If we make the substitution $u = g(x)$, then

$$\begin{aligned} \int f(g(x))g'(x)dx &= \int \frac{d}{dx} F(g(x))dx \\ &= F(g(x)) + C \\ &= F(u) + C \\ &\int F'(u)du \\ &= \int f(u)du. \end{aligned}$$

1.15 Theorem 8 - Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Proof Let F denote any antiderivative of f. Then,

$$\int_a^b f(g(x)) \cdot g'(x)dx = F(x)\Big]_{x=a}^{x=b}$$

$$= F(g(b)) - F(g(a))$$

$$= F(u)\Big]_{u=g(a)}^{u=g(b)}$$

$$\int_{g(a)}^{g(b)} f(u)du.$$

1.16 Theorem 9

Let f be continuous on the symmetric interval $[-a, a]$.

1. If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.
2. if f is odd, then $\int_{-a}^{a} f(x)dx = 0$.

Proof of Part (a)

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx.$$

$$= -\int_0^{-a} f(x)dx + \int_0^a f(x)dx$$

$$= -\int_0^a f(-u)(-du) + \int_0^a f(x)dx$$

$$= \int_0^a f(-u)du + \int_0^a f(x)dx$$

$$= \int_0^a f(u)du + \int_0^a f(x)dx$$

$$= 2\int_0^a f(x)dx$$

Proof for part 2 is entirely similar. Thus, we leave it for the reader to complete.

1.17 Area of a region between two curves

if f and g are continuous with $f(x) \leq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]dx.$$